

# Discrete Ramanujan-Fourier Transform of Even Functions (mod $r$ )

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**Abstract.** An arithmetical function  $f$  is said to be even (mod  $r$ ) if  $f(n) = f((n, r))$  for all  $n \in \mathbb{Z}^+$ , where  $(n, r)$  is the greatest common divisor of  $n$  and  $r$ . We adopt a linear algebraic approach to show that the Discrete Fourier Transform of an even function (mod  $r$ ) can be written in terms of Ramanujan's sum and may thus be referred to as the Discrete Ramanujan-Fourier Transform.

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## 1 Introduction

By an arithmetical function we mean a complex-valued function defined on the set of positive integers. For a positive integer  $r$ , an arithmetical function  $f$  is said to be *periodic* (mod  $r$ ) if  $f(n + r) = f(n)$  for all  $n \in \mathbb{Z}^+$ . Every periodic function  $f$  (mod  $r$ ) can be written uniquely as

$$f(n) = r^{-1} \sum_{k=1}^r F_f(k) \epsilon_k(n), \quad (1)$$

where

$$F_f(k) = \sum_{n=1}^r f(n)\epsilon_k(-n) \quad (2)$$

and  $\epsilon_k$  denotes the periodic function  $\pmod{r}$  defined as

$$\epsilon_k(n) = \exp(2\pi i kn/r).$$

The function  $F_f$  in (2) is referred to as the Discrete Fourier Transform (DFT) of  $f$ , and (1) is the Inverse Discrete Fourier Transform (IDFT).

An arithmetical function  $f$  is said to be *even*  $\pmod{r}$  if

$$f(n) = f((n, r))$$

for all  $n \in \mathbb{Z}^+$ , where  $(n, r)$  is the greatest common divisor of  $n$  and  $r$ . It is easy to see that every even function  $\pmod{r}$  is periodic  $\pmod{r}$ . Ramanujan's sum  $C(n, r)$  is defined as

$$C(n, r) = \sum_{\substack{k \pmod{r} \\ (k, r)=1}} \exp(2\pi i kn/r)$$

and is an example of an even function  $\pmod{r}$ .

In this paper we show that the DFT (2) and IDFT (1) of an even function  $f \pmod{r}$  can be written in a concise form using Ramanujan's sum  $C(n, r)$ , see Section 3. We also review a proof of (1) and (2) for periodic functions  $\pmod{r}$ , see Section 2, and review (1) and (2) for the Cauchy product of periodic functions  $\pmod{r}$ , see Section 4. The Cauchy product of periodic functions  $f$  and  $g \pmod{r}$  is defined as

$$(f \circ g)(n) = \sum_{a+b \equiv n \pmod{r}} f(a)g(b).$$

The results of this paper may be considered to be known. They have not been presented in exactly this form and we hope that this paper will provide a clear approach to the elementary theory of even functions  $\pmod{r}$ .

The concept of an even function  $\pmod{r}$  originates from Cohen [2] and was further studied by Cohen in subsequent papers [3, 4, 5]. General accounts of even functions  $\pmod{r}$  can be found in the books by McCarthy [8] and Sivaramakrishnan [10]. For recent papers on even functions  $\pmod{r}$  we refer to [9, 11]. Material on periodic functions  $\pmod{r}$  can be found in the book by Apostol [1].

## 2 Proof of (1) and (2)

Let  $P_r$  denote the set of all periodic arithmetical functions  $(\bmod r)$ . It is clear that  $P_r$  is a complex vector space under the usual addition and scalar multiplication. In fact,  $P_r$  is isomorphic to  $\mathbb{C}^r$ . Further,  $P_r$  is a complex inner product space under the Euclidean inner product given as

$$\langle f, g \rangle = \sum_{n=1}^r f(n) \overline{g(n)} = (f\overline{g} \circ \zeta)(r), \quad (3)$$

where  $\zeta$  is the constant function 1. The set  $\{r^{-1/2}\epsilon_k : k = 1, 2, \dots, r\}$  is an orthonormal basis of  $P_r$ . Thus, every  $f \in P_r$  can be written uniquely as

$$f(n) = \sum_{k=1}^r \langle f, r^{-1/2}\epsilon_k \rangle r^{-1/2}\epsilon_k(n),$$

where

$$\langle f, r^{-1/2}\epsilon_k \rangle = \sum_{n=1}^r f(n) \overline{r^{-1/2}\epsilon_k(n)} = r^{-1/2} \sum_{n=1}^r f(n) \epsilon_k(-n).$$

This proves (1) and (2).

## 3 DFT and IDFT for even functions $(\bmod r)$

Let  $E_r$  denote the set of all even functions  $(\bmod r)$ . The set  $E_r$  forms a complex vector space under the usual addition and scalar multiplication. In fact,  $E_r$  is a subspace of  $P_r$ . Thus (1) and (2) hold for  $f \in E_r$ . We can also present (1) and (2) for  $f \in E_r$  in terms of Ramanujan's sum as is shown below.

Note that Ramanujan's sum  $C(n, r)$  is an integer for all  $n$  and can be evaluated by addition and subtraction of integers. In fact,  $C(n, r)$  can be written as  $C(n, r) = \sum_{d|(n, r)} d\mu(r/d)$ , where  $\mu$  is the Möbius function.

An arithmetical function  $f \in E_r$  is completely determined by its values  $f(d)$  with  $d|r$ . Thus  $E_r$  is isomorphic to  $\mathbb{C}^{\tau(r)}$ , where  $\tau(r)$  is the number of divisors of  $r$ . The inner product (3) in  $P_r$  can be written in  $E_r$  in terms of the Dirichlet convolution. In fact, we have

$$\sum_{\substack{k=1 \\ (k, r)=d}}^r 1 = \sum_{\substack{j=1 \\ (j, r/d)=1}}^{r/d} 1 = \phi(r/d), \quad (4)$$

where  $\phi$  is Euler's totient function, and thus (3) can be written for  $f, g \in E_r$  as

$$\langle f, g \rangle = \sum_{k=1}^r f(k) \overline{g(k)} = \sum_{d|r} f(d) \overline{g(d)} \phi(r/d) = (f \bar{g} * \phi)(r),$$

where  $*$  is the Dirichlet convolution.

**Theorem 3.1.** *The set*

$$\{(r\phi(d))^{-\frac{1}{2}} C(\cdot, d) : d \mid r\} \quad (5)$$

*is an orthonormal basis of the inner product space  $E_r$ .*

*Proof* As the dimension of the inner product space  $E_r$  is  $\tau(r)$  and the number of elements in the set (5) is  $\tau(r)$ , it suffices to show the set (5) is an orthonormal subset of  $E_r$ . This follows easily from the relation

$$\sum_{e|r} C(r/e, d_1) C(r/e, d_2) \phi(e) = \begin{cases} r\phi(d_1) & \text{if } d_1 = d_2, \\ 0 & \text{otherwise,} \end{cases}$$

where  $d_1 \mid r$  ja  $d_2 \mid r$  (see [8, p. 79]).  $\square$

We now present (1) and (2) for  $f \in E_r$ .

**Theorem 3.2.** *Every  $f \in E_r$  can be written uniquely as*

$$f(n) = r^{-1} \sum_{d|r} R_f(d) C(n, d), \quad (6)$$

*where*

$$R_f(d) = \phi(d)^{-1} \sum_{n=1}^r f(n) C(n, d). \quad (7)$$

*Proof* On the basis of Theorem 3.1,

$$f(n) = \sum_{d|r} \langle f, (r\phi(d))^{-\frac{1}{2}} C(\cdot, d) \rangle (r\phi(d))^{-\frac{1}{2}} C(n, d). \quad (8)$$

Applying (3) to (8) we obtain (6) and (7).  $\square$

The function  $R_f$  in (7) may be referred to as the Discrete Ramanujan-Fourier Transform of  $f$ , and (6) may be referred to as the Inverse Discrete Ramanujan-Fourier Transform. Cf. [8].

Another expression of (7) can be obtained easily. Namely, applying (4) to (8) and then applying

$$\phi(e) C(r/e, d) = \phi(d) C(r/d, e)$$

(see [8, p. 93]) we obtain

$$R_f(d) = \sum_{e|r} f(r/e)C(r/d, e). \quad (9)$$

Note that (6) can also be derived from (1). In fact, if  $f \in E_r$ , then (2) can be written as

$$\begin{aligned} F_f(k) &= \sum_{n=1}^r f(n) \exp(-2\pi i k n / r) \\ &= \sum_{e|r} \sum_{\substack{n=1 \\ (n,r)=e}}^r f(e) \exp(-2\pi i k n / r) \\ &= \sum_{e|r} f(e) \sum_{\substack{m=1 \\ (m,r/e)=1}}^{r/e} \exp(-2\pi i k m / (r/e)) \\ &= \sum_{e|r} f(e) C(k, r/e). \end{aligned}$$

A similar argument shows (6) with  $R_f(d) = F_f(r/d)$ . We omit the details.

## 4 The Cauchy product

It is well known that if  $h$  is the Cauchy product of  $f \in P_r$  and  $g \in P_r$ , then  $F_h = F_f F_g$ . This follows from the property

$$\sum_{a+b \equiv n \pmod{r}} \epsilon_k(a) \epsilon_j(b) = \begin{cases} r \epsilon_k(n) & \text{if } k \equiv j \pmod{r}, \\ 0 & \text{otherwise.} \end{cases}$$

Analogously, if  $h$  is the Cauchy product of  $f \in E_r$  and  $g \in E_r$ , then  $R_h = R_f R_g$ . This follows from the property

$$\sum_{a+b \equiv n \pmod{r}} C(a, d_1) C(b, d_2) = \begin{cases} r C(a, d_1) & \text{if } d_1 = d_2, \\ 0 & \text{otherwise,} \end{cases}$$

where  $d_1 \mid r$  ja  $d_2 \mid r$  (see [10, p. 333]).

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